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Analysis of velocity fluctuation in turbulence based on generalized statistics

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Abstract

The numerical turbulence experiments conducted by Gotoh *et al* are analysed with high precision with the help of formulae for the scaling exponents of the velocity structure function and for the probability density function of the velocity fluctuations. These formulae are derived by the present authors, with the multifractal aspect based on statistics constructed using generalized measures of entropy, i.e., the extensive Rényi entropy or the non-extensive Tsallis entropy. It is shown, explicitly, that there exist *two* scaling regions, i.e., the *upper* scaling region with larger separations which may correspond to the scaling range observed by Gotoh *et al* and the *lower* scaling region with smaller separations which is a new scaling region, extracted for the first time by the present systematic analysis. These scaling regions are divided by a definite length approximately of the order of the Taylor microscale, which may correspond to the crossover length proposed by Gotoh *et al* as the low end of the scaling range (i.e., the *upper* scaling region).

Since the discovery of the Kolmogorov spectrum [1], there have been many investigations, including those in [2, 3], carried out in an effort to understand the intermittent evolution of fluid in fully developed turbulence. Among them, the line of study based on a kind of *ensemble*-theoretical approach, starting from the log-normal model [4–6], continues with the β-model (a one-fractal dimensional analysis) [7], the p-model (a multifractal model) [8, 9], the 3D binomial Cantor set model [10] and so on. Among this series, an investigation of turbulence based on generalized entropy, i.e., the Rényi [11] entropy or the Tsallis [12, 13] entropy, was initiated by the present authors [14–20]. After a rather limited investigation of the p-model [14], the approach was developed further in order to permit derivation of the analytical expression for the scaling exponents of the velocity structure function [15–18] and determination of the probability density function (PDF) of the velocity fluctuations [18–20] by a self-consistent statistical mechanical approach.

With the help of the analytical formulae derived in [15–20], we will analyse, in this paper, the PDFs of velocity fluctuations observed in the beautiful DNS (i.e., the direct numerical simulation) conducted by Gotoh *et al* [21]. We will deal with the data for the Taylor microscale Reynolds number $R_\lambda = 381$, since for this Reynolds number Gotoh *et al* obtained the PDF with accuracy up to the order of 10^{-9}–10^{-10}—far better than that in any previous experiments, real or numerical. We showed in [19] that our formulae can explain quite well the PDFs observed in the real experiment by Lewis and Swinney [22] for turbulent Couette–Taylor flow at $R_\lambda = 270$ ($Re = 5.4 \times 10^5$) produced in a concentric cylinder system in which, however, the PDFs were measured only with accuracy of the order of 10^{-5}. Note that the success of the present theory in the analysis of this turbulent Couette–Taylor flow may indicate the robustness of singularities associated with the velocity gradient even for the case of no inertial range. We have already made clear in [20] that the present theory can also explain quite well the PDFs of *longitudinal* velocity fluctuations reported by Gotoh *et al* [21], and have revealed the superiority of our PDF to the one derived in [23] when the accuracy is raised to of the order of 10^{-9}. Encouraged by the success of these preliminary tests, we will apply our theory to make further precise analyses of the data obtained in [21], including the PDFs of *transverse* velocity fluctuations in addition to those of *longitudinal* fluctuations.

The basic equation describing fully developed turbulence is the Navier–Stokes equation $\partial\vec{u}/\partial t + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}(p/\rho) + \nu\,\nabla^2\vec{u}$ for an incompressible fluid, where ρ, p and ν represent, respectively, the mass density, the pressure and the kinematic viscosity. Our main interest is in the correlation of measured time series for the streamwise velocity component u, say the x-component of the fluid velocity field $\vec{u}$, in the turbulent flow produced by a grid with size ℓ_0 put in a laminar flow. Applying Taylor's frozen-flow hypothesis, the quantity of interest reduces to $\delta u(r) = |u(x + r) - u(x)|$ representing the spatial change of the component u. We assume that downstream of the grid there appear a cascade of eddies with different sizes $\ell_n = \delta_n \ell_0$ where $\delta_n = \delta^{-n}$ ($\delta > 1, n = 0, 1, 2, \ldots$). At each step of the cascade, say at the nth step, eddies break up into δ pieces, producing an energy cascade with the energy-transfer rate ϵ_n that represents the rate of transfer of energy per unit mass from eddies of size ℓ_n to ones of size ℓ_{n+1}.

The Reynolds number Re of the system is given by $Re = \delta u_0\, \ell_0/\nu = (\ell_0/\eta)^{4/3}$ where $\eta = (\nu^3/\epsilon)^{1/4}$ is the Kolmogorov scale [1] with ϵ the rate of energy input to the largest eddies of size ℓ_0, i.e., $\epsilon_0 = \epsilon$. Here, we have introduced the notation $\delta u_n = \delta u(\ell_n)$ representing the velocity difference across a distance $r \sim \ell_n$. Thus, our main focus of interest in the following reduces to the fluctuation of the velocity difference δu_n corresponding to the size of the nth eddy in the cascade. Note that the dependence of the number of steps n on r/η, within the analysis where intermittency is not taken into account [1], is given by

$$n = -\log_\delta r/\eta + (3/4)\log_\delta Re. \tag{1}$$

For homogeneous and isotropic turbulence, there is a relation between the Taylor microscale Reynolds number R_λ and the Reynolds number Re [24]: $R_\lambda^2 = A\, Re$ where A is a real number of the order of 0.01–10 depending on the experimental set-up.

For high Reynolds number $Re \gg 1$ or for the situation where the effects of the kinematic viscosity can be neglected compared with those of the turbulent viscosity, the Navier–Stokes equation for incompressible fluid is invariant under the scale transformations [9, 25] $\vec{r} \to \lambda\vec{r}$, $\vec{u} \to \lambda^{\alpha/3}\vec{u}$, $t \to \lambda^{1-\alpha/3}t$ and $(p/\rho) \to \lambda^{2\alpha/3}(p/\rho)$. Here, the exponent α is an arbitrary real quantity which specifies the degree of singularity in the velocity gradient $|\partial u(x)/\partial x| = \lim_{\ell_n\to 0} |u(x + \ell_n) - u(x)|/\ell_n = \lim_{\ell_n\to 0} \delta u_n/\ell_n$. This can be seen from the relation $\delta u_n/\delta u_0 = (\ell_n/\ell_0)^{\alpha/3}$, which leads to the singularity in the velocity gradient [26] for $\alpha < 3$, since $\delta u_n/\ell_n^{\alpha/3} = \text{constant}$. We also have the relation $\epsilon_n/\epsilon = (\ell_n/\ell_0)^{\alpha-1}$.

Let us determine the probability $P^{(n)}(\alpha)\,\mathrm{d}\alpha$ of finding at a point in physical space an eddy of size ℓ_n which has a value of the degree of singularity in the range $\alpha \sim \alpha + \mathrm{d}\alpha$.

Assuming that each step in the cascade is statistically independent, i.e., $P^{(n)}(\alpha) = [P^{(1)}(\alpha)]^n$, our task reduces to determining $P^{(1)}(\alpha)$. In order to proceed, we have assumed [15–20] that the underlying statistics describing the intermittent evolution of fully developed turbulence is the one given by the Rényi entropy [11] $S_q^{\mathrm{R}}[P^{(1)}(\alpha)] = (1-q)^{-1} \ln \int \mathrm{d}\alpha\, P^{(1)}(\alpha)^q$, or that given by the Tsallis entropy [12, 13, 27] $S_q^{\mathrm{T}}[P^{(1)}(\alpha)] = (1-q)^{-1} \left(\int \mathrm{d}\alpha\, P^{(1)}(\alpha)^q - 1 \right)$. Taking an extremum of each of these entropies with appropriate constraints, i.e., the normalization of the distribution function $\int \mathrm{d}\alpha\, P^{(1)}(\alpha) =$ constant and the q-variance kept constant as a known quantity: $\sigma_q^2 = \langle (\alpha - \alpha_0)^2 \rangle_q = \left(\int \mathrm{d}\alpha\, P^{(1)}(\alpha)^q (\alpha - \alpha_0)^2 \right) / \int \mathrm{d}\alpha\, P^{(1)}(\alpha)^q$, we have [15–18]

$$P^{(1)}(\alpha) \propto \left[1 - (\alpha - \alpha_0)^2 / (\Delta\alpha)^2 \right]^{1/(1-q)} \tag{2}$$

with $(\Delta\alpha)^2 = 2X/[(1-q)\ln 2]$. The Rényi entropy has an information-theoretical basis, and has extensive character, as the usual thermodynamical entropy does. On the other hand, the Tsallis entropy is non-extensive, and therefore it provides us with an attractive test for the generalized statistical mechanics which deals with, for example, systems with long-range correlations with a hierarchical structure where the usual extensive characteristics of statistical mechanics are not present. Note that the distribution functions which give an extremum of each entropy have a common structure (see (2) for the present system where $q \leqslant 1$), in spite of the different characteristics of these entropies, and that the values of α are restricted to within the range $[\alpha_{\min}, \alpha_{\max}]$, where $\alpha_{\max} - \alpha_0 = \alpha_0 - \alpha_{\min} = \Delta\alpha$. Note that $\sigma_q^2 = 2X/[(3-q)\ln 2]$.

By making use of an observed value of the intermittency exponent μ as an input, the quantities α_0, X and the index q can be determined, self-consistently, with the help of the three independent equations, i.e., the energy conservation: $\langle \epsilon_n \rangle = \epsilon$; the definition of the intermittency exponent μ: $\langle \epsilon_n^2 \rangle = \epsilon^2 \delta_n^{-\mu}$; and the scaling relation: $1/(1-q) = 1/\alpha_- - 1/\alpha_+$ with $\alpha_\pm$ satisfying $f(\alpha_\pm) = 0$ [15–18]. Here, the average $\langle \cdots \rangle$ is taken with $P^{(n)}(\alpha)$. The scaling relation is a generalization of the one derived first in [28, 29] to the case where the multifractal spectrum has negative values. For the region where the value of μ is usually observed, i.e., $0.13 \leqslant \mu \leqslant 0.40$, the three self-consistent equations are solved to give the approximate equations $\alpha_0 = 0.9989 + 0.5814\mu$, $X = -2.848 \times 10^{-3} + 1.198\mu$ and $q = -1.507 + 20.58\mu - 97.11\mu^2 + 260.4\mu^3 - 365.4\mu^4 + 208.3\mu^5$. These equations are slightly different from those given in [19], since the region of μ has been extended a little bit.

If we put $q = 1$ from the beginning, i.e., start with the Boltzmann–Shannon entropy, and take an extremum with the same constraints as were given above for $q = 1$, we have a Gaussian distribution function. The parameters α_0 and X are then determined by two conditions, i.e., energy conservation and the definition of the intermittency exponent. The results obtained [18] completely coincide with those in the log-normal model [4–6]. Therefore, the present approach can be interpreted as an extension of the one in the log-normal model.

It is known that in turbulent flow there are two mechanisms that govern its dissipative evolution, i.e., the one controlled by the kinematic viscosity that is responsible for thermal fluctuations and the one controlled by the turbulent viscosity that is responsible for intermittent fluctuations related to the singularities in the velocity gradient. Therefore, it may be reasonable to assume that the probability $\Pi^{(n)}(x_n)\,\mathrm{d}x_n$ of finding the scaled velocity fluctuation $|x_n| = \delta u_n / \delta u_0$ in the range $x_n \sim x_n + \mathrm{d}x_n$ can be divided into two parts:

$$\Pi^{(n)}(x_n)\,\mathrm{d}x_n = \Pi_{\mathrm{N}}^{(n)}(x_n)\,\mathrm{d}x_n + \Pi_{\mathrm{S}}^{(n)}(|x_n|)\,\mathrm{d}x_n. \tag{3}$$

Here, the normal-part PDF $\Pi_{\mathrm{N}}^{(n)}(x_n)$ stems from thermal dissipation and the singular-part PDF $\Pi_{\mathrm{S}}^{(n)}(|x_n|)$ stems from multifractal distribution of the singularities. The latter is derived through $\Pi_{\mathrm{S}}^{(n)}(|x_n|)\,\mathrm{d}x_n = P^{(n)}(\alpha)\,\mathrm{d}\alpha$ with the following transformation of the variables: $|x_n| = \delta_n^{\alpha/3}$.

The mth moments of the velocity fluctuations, defined by $\langle\langle |x_n|^m \rangle\rangle = \int_{-\infty}^{\infty} \mathrm{d}x_n\, |x_n|^m \Pi^{(n)}(x_n)$, are given by

$$\langle\langle |x_n|^m \rangle\rangle = 2\gamma_m^{(n)} + (1 - 2\gamma_0^{(n)}) a_m \delta_n^{\zeta_m} \tag{4}$$

where $a_{3\bar{q}} = \{2/[C_{\bar{q}}^{1/2}(1 + C_{\bar{q}}^{1/2})]\}^{1/2}$, with $C_{\bar{q}} = 1 + 2\bar{q}^2(1-q)X \ln 2$, and

$$2\gamma_m^{(n)} = \int_{-\infty}^{\infty} \mathrm{d}x_n\, |x_n|^m \Pi_{\mathrm{N}}^{(n)}(x_n). \tag{5}$$

We used the normalization $\langle\langle 1 \rangle\rangle = 1$. The quantity

$$\zeta_m = \alpha_0 m/3 - 2Xm^2/[9(1 + C_{m/3}^{1/2})] - [1 - \log_2(1 + C_{m/3}^{1/2})]/(1-q) \tag{6}$$

is the so-called scaling exponent of the velocity structure function, whose expression was first derived by the present authors [15–18]. In this paper, we will use the formula (6) in order to extract the value of the intermittency exponent μ for the best fit to the measured scaling exponents by the method of least squares.

There usually appears an asymmetry (a negative skewness) in the PDFs of the *longitudinal* velocity fluctuations, but this is not so clear in the PDFs of the *transverse* ones (see, for example, figure 15 and 16 in [21]). Assuming that the singularities in the velocity gradient contribute mainly to the symmetric part, we will deal with the symmetric part of the PDFs of the velocity fluctuations in the following:

With the help of the new variable

$$\xi_n = \delta u_n / \langle\langle \delta u_n^2 \rangle\rangle^{1/2} = x_n / \langle\langle x_n^2 \rangle\rangle^{1/2} = \bar{\xi}_n \delta_n^{\alpha/3 - \zeta_2/2} \tag{7}$$

scaled by the variance of velocity fluctuations, the PDF $\hat{\Pi}^{(n)}(|\xi_n|)$, introduced through $\hat{\Pi}^{(n)}(|\xi_n|)\, \mathrm{d}\xi_n = \Pi^{(n)}(|x_n|)\, \mathrm{d}x_n$, is given by [19, 20]

$$\hat{\Pi}^{(n)}(\xi_n) = \hat{\Pi}_{<*}^{(n)}(\xi_n) \qquad \text{for } |\xi_n| \leqslant \xi_n^* \tag{8}$$

$$\hat{\Pi}^{(n)}(\xi_n) = \hat{\Pi}_{*<}^{(n)}(\xi_n) \qquad \text{for } \xi_n^* \leqslant |\xi_n| \leqslant \bar{\xi}_n \delta_n^{\alpha_{\min}/3 - \zeta_2/2}. \tag{9}$$

Here, $\bar{\xi}_n = [2\gamma_2^{(n)} \delta_n^{-\zeta_2} + (1 - 2\gamma_0^{(n)}) a_2]^{-1/2}$. Assuming that, for smaller velocity fluctuations $|\xi_n| \lesssim \xi_n^*$, the contribution to the PDF of the velocity fluctuations comes mainly from thermal fluctuations related to the kinematic viscosity, we take for the PDF $\hat{\Pi}_{<*}^{(n)}(\xi_n)$ a Gaussian function [19, 20], i.e.,

$$\hat{\Pi}_{<*}^{(n)}(\xi_n) = \bar{\Pi}_{\mathrm{S}}^{(n)} \mathrm{e}^{-[1 + 3f'(\alpha^*)][(\xi_n/\xi_n^*)^2 - 1]/2} \tag{10}$$

with $\bar{\Pi}_{\mathrm{S}}^{(n)} = 3(1 - 2\gamma_0^{(n)})/(2\bar{\xi}_n \sqrt{2\pi X |\ln \delta_n|})$. On the other hand, we assume that the main contribution to $\hat{\Pi}_{*<}^{(n)}(\xi_n)$ may come from the multifractal distribution of singularities [18–20] related to the turbulent viscosity, i.e., $\hat{\Pi}_{*<}^{(n)}(\xi_n) = \hat{\Pi}_{\mathrm{S}}^{(n)}(|\xi_n|)$:

$$\hat{\Pi}_{*<}^{(n)}(\xi_n) = \bar{\Pi}_{\mathrm{S}}^{(n)} \frac{\bar{\xi}_n}{|\xi_n|} \left[1 - \left(\frac{3 \ln |\xi_n / \xi_{n,0}|}{\Delta\alpha\, |\ln \delta_n|} \right)^2 \right]^{n/(1-q)} \tag{11}$$

with $|\xi_{n,0}| = \bar{\xi}_n \delta_n^{\alpha_0/3 - \zeta_2/2}$. The point ξ_n^* was defined by $\xi_n^* = \bar{\xi}_n \delta_n^{\alpha^*/3 - \zeta_2/2}$ where α^* is the solution of $\zeta_2/2 - \alpha/3 + 1 - f(\alpha) = 0$, and $\hat{\Pi}_{<*}^{(n)}(\xi_n)$ and $\hat{\Pi}_{*<}^{(n)}(\xi_n)$ were connected at ξ_n^* under the condition that they should have the same value and the same derivative there. Here,

$$f(\alpha) = 1 + (1-q)^{-1} \log_2[1 - (\alpha - \alpha_0)^2/(\Delta\alpha)^2] \tag{12}$$

is the multifractal spectrum [15–18], derived from the relation $P^{(n)}(\alpha) \propto \delta_n^{1 - f(\alpha)}$ [9, 18], that reveals how densely each singularity, labelled by α, fills physical space.

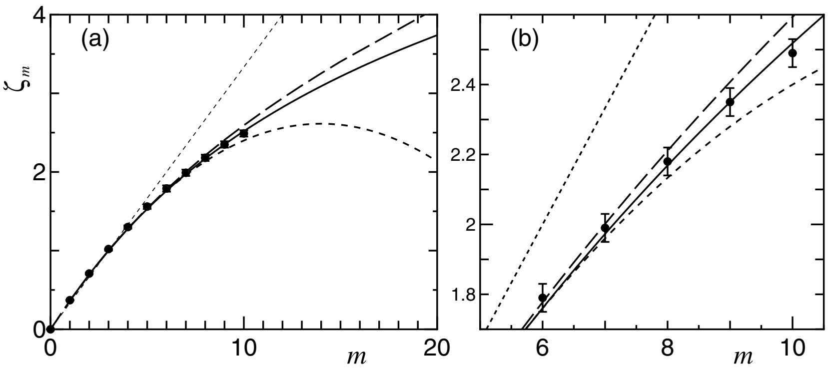

Figure 1. The *longitudinal* scaling exponents $\zeta_m = \zeta_m^{\mathrm{L}}$ of the velocity structure function. Panel (b) is a magnification of the panel (a). The results of DNS obtained by Gotoh *et al* ($R_\lambda = 381$) are shown by closed circles. The present theoretical formula (6) was used to determine the value $\mu = 0.240$ of the intermittency exponent, and the result is shown by solid curves. Dotted lines represent the results of Kolmogorov (K41); dashed lines represent She–Leveque results. The prediction of the log-normal model is given by short-dashed lines with $\mu = 0.240$.

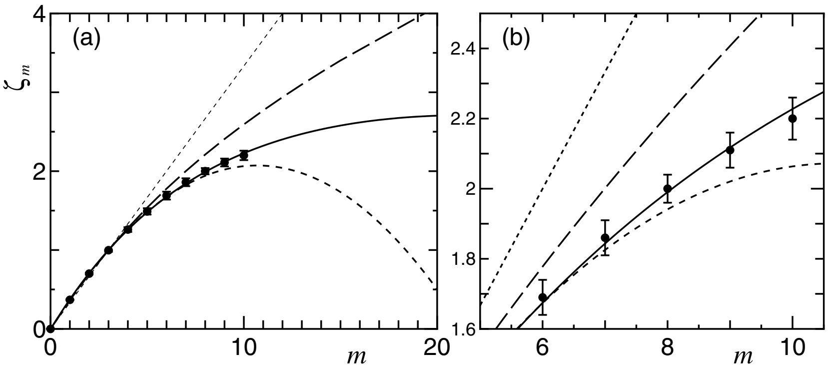

Figure 2. The *transverse* scaling exponents $\zeta_m = \zeta_m^{\mathrm{T}}$ of the velocity structure function. Panel (b) is a magnification of the panel (a). The results of DNS obtained by Gotoh *et al* ($R_\lambda = 381$) are shown by closed circles. The present theoretical formula (6) was used to determine the value $\mu = 0.327$ of the intermittency exponent, and the result is shown by solid lines. Dotted lines represent K41; dashed lines represent She–Leveque results. The prediction of the log-normal model is given by short-dashed lines with $\mu = 0.327$.

Now, we proceed to analyse the data from the DNS conducted by Gotoh *et al* [21] for fully developed turbulence. In the following theoretical analyses, we will take $\delta = 2$ for the number of pieces of 'eddies' generated at each step in the energy cascade. In figures 1 and 2, we put, respectively, the measured scaling exponents ζ_m^{L} for longitudinal velocity fluctuations and ζ_m^{T} for transverse ones at $R_\lambda = 381$ (closed circles) [21]. The scaling exponents (6) derived from the present theory are given by a solid curve in each figure. There are also represented, for reference, the predictions of K41 (dotted curve) [1], of She–Leveque (dashed curve) [30] and of the log-normal model (short-dashed curve) [4–6]. We determine, self-consistently, the values of the intermittency exponent μ by fitting the formula (6) with the ten observed values of the scaling exponents, ζ_m^{L} and ζ_m^{T}, with the help of the method of least squares. The values determined for μ are listed in table 1, with the values of the corresponding parameters q, α_0 and X. The latter three parameters are obtained from the μ-dependent functions that are given in this paper

Table 1. Comparison of the values of the scaling exponents (6) derived using the present theory and those for the *longitudinal* and *transverse* velocity fluctuations observed in DNS conducted by Gotoh *et al*. The value μ for each fluctuation is also listed, with the corresponding values for q, α_0 and X.

	Longitudinal		Transverse	
μ	0.240		0.327	
q	0.391		0.543	
α_0	1.138		1.189	
X	0.285		0.388	
	ζ_m^{L}		ζ_m^{T}	
m	Present theory	DNS	Present theory	DNS
1	0.3637	0.370 ± 0.004	0.3747	0.369 ± 0.004
2	0.6965	0.709 ± 0.009	0.7073	0.701 ± 0.01
3	1.000	1.02 ± 0.02	1.000	0.998 ± 0.02
4	1.277	1.30 ± 0.02	1.256	1.26 ± 0.03
5	1.529	1.56 ± 0.03	1.480	1.49 ± 0.04
6	1.761	1.79 ± 0.04	1.674	1.69 ± 0.05
7	1.973	1.99 ± 0.04	1.843	1.86 ± 0.05
8	2.169	2.18 ± 0.04	1.990	2.00 ± 0.04
9	2.350	2.35 ± 0.04	2.117	2.11 ± 0.05
10	2.519	2.49 ± 0.04	2.227	2.20 ± 0.06

as the solutions of the self-consistent equations. Note that the theoretical lines thus determined are located within the experimental error bars at each of the ten observed points. Therefore, the relation $\mu = 2 - \zeta_6$ is satisfied within the experimental error bars. We obtain $\alpha_+ - \alpha_0 = \alpha_0 - \alpha_- = 0.6818\,(0.8167)$, $\Delta\alpha = 1.160\,(1.566)$ for the longitudinal (transverse) fluctuations.

There is an argument that the scaling exponents ζ_2^{L} for the longitudinal velocity structure function and ζ_2^{T} for the transverse function should be equal for isotropic and incompressible turbulence. In table 1, we see that $\zeta_2^{\mathrm{L}} = 0.696$ and $\zeta_2^{\mathrm{T}} = 0.707$ within the present theoretical analysis, which give, respectively, 1.696 and 1.707 for the exponent of the Kolmogorov spectrum which is $5/3 = 1.\dot{6}$ in K41 [1]. The small deviation (1%) between ζ_2^{L} and ζ_2^{T} can be attributed to the finite sample size and the small amount of flow anisotropy [21]. The scaling exponents ζ_4 of the fourth-order moment, related to the second-order pressure structure function, are reported in [21] as $\zeta_4^{\mathrm{L}} = 1.30 \pm 0.02$ and $\zeta_4^{\mathrm{T}} = 1.26 \pm 0.03$ at $R_\lambda = 381$. These values are comparable with the ones in table 1, i.e., $\zeta_4^{\mathrm{L}} = 1.277$ and $\zeta_4^{\mathrm{T}} = 1.256$ derived using the present theory.

The comparison between the measured PDFs of the velocity fluctuations at $R_\lambda = 381$ obtained by DNS [21] and those obtained by the present analysis [18–20] is shown in figure 3 for longitudinal fluctuations and in figure 4 for transverse ones. In order to extract the symmetrical part, we took averages of the left- and right-hand side DNS data. The symmetrized data are shown by closed circles. The solid lines are the curves $\hat{\Pi}^{(n)}(\xi_n)$ given by (10) and (11) with the values of the parameters in table 1. The values of ξ_n^* are about 1–1.5. The separations $r/\eta = \ell_n/\eta$ are, from top to bottom: 2.38, 4.76, 9.52, 19.0, 38.1, 76.2, 152, 305, 609 and 1220 for DNS [21] data. On the other hand, the numbers n of steps in the cascade for longitudinal (transverse) fluctuations are, from top to bottom: 21.5, 20.0, 16.8, 14.0, 11.8, 10.1, 9.30, 8.10, 7.00 and 6.00 (20.0, 18.2, 14.6, 11.4, 9.30, 7.90, 6.80, 5.60, 4.70 and 4.00). These values are obtained by the method of least squares with respect to the logarithm of the PDFs for the best fit of our theoretical formulae (10) and (11) to the observed values of the

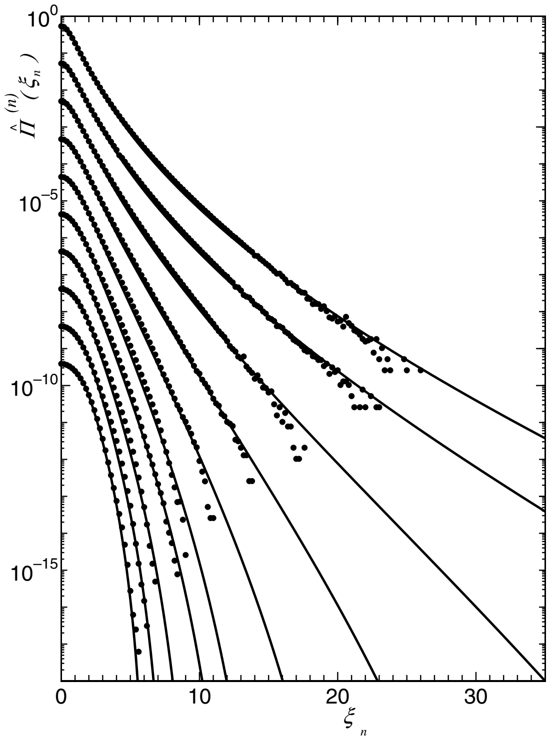

Figure 3. The experimental PDFs of the *longitudinal* velocity fluctuations measured by Gotoh *et al* for $R_\lambda = 381$ are compared with the present theoretical results $\hat{\Pi}^{(n)}(\xi_n)$. Closed circles show the symmetrized points obtained by taking averages of the left- and the right-hand-side DNS data. For the experimental data, the separations $r/\eta = \ell_n/\eta$ are, from top to bottom: 2.38, 4.76, 9.52, 19.0, 38.1, 76.2, 152, 305, 609 and 1220. Solid lines represent the curves given by the present theory with $q = 0.391$ ($\mu = 0.240$). For the theoretical curves, the numbers of steps in the cascade n are, from top to bottom: 21.5, 20.0, 16.8, 14.0, 11.8, 10.1, 9.30, 8.10, 7.00 and 6.00. For better visibility, each PDF is shifted by -1 unit along the vertical axis.

PDFs, discarding those points which have observed values less than 10^{-9} (10^{-8}) since they show substantial scatter on the logarithmic scale. We see an excellent agreement between the measured PDFs and the analytical formula for the PDF derived from the present self-consistent theory.

The dependence of n on r/η for longitudinal (transverse) fluctuations, extracted from figure 3 (figure 4), is shown in figure 5(a) (figure 5 (b)) by solid and dashed curves. These curves are obtained by the method of least squares within linear fits, and are given by

$$n = -1.050 \log_2 r/\eta + 16.74 \qquad (\text{for } \ell_c^{\mathrm{L}} \leqslant r) \tag{13}$$

$$n = -2.540 \log_2 r/\eta + 25.08 \qquad (\text{for } r < \ell_c^{\mathrm{L}}) \tag{14}$$

with the crossover length $\ell_c^{\mathrm{L}}/\eta = 48.26$ for longitudinal fluctuations and

$$n = -0.9896 \log_2 r/\eta + 13.95 \qquad (\text{for } \ell_c^{\mathrm{T}} \leqslant r) \tag{15}$$

$$n = -2.820 \log_2 r/\eta + 23.87 \qquad (\text{for } r < \ell_c^{\mathrm{T}}) \tag{16}$$

with the crossover length $\ell_c^{\mathrm{T}}/\eta = 42.57$ for transverse fluctuations. We see that $\ell_c^{\mathrm{T}}/\eta \lesssim \ell_c^{\mathrm{L}}/\eta$, and that these crossover lengths have values close to the reported value of the Taylor microscale $\lambda/\eta = 38.33$ in [21] at $R_\lambda = 381$. The value of ℓ_c^{T}/η is very close to the one in figure 31

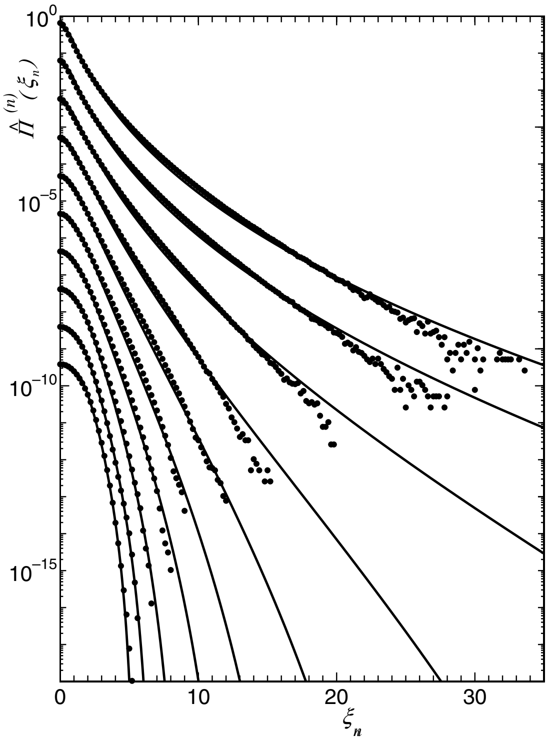

Figure 4. The experimental PDFs of the *transverse* velocity fluctuations measured by Gotoh *et al* for $R_\lambda = 381$ are compared with the present theoretical results $\hat{\Pi}^{(n)}(\xi_n)$. Closed circles show the symmetrized points. The separations $r/\eta = \ell_n/\eta$ are the same as in figure 3. Solid lines represent the curves given by the present theory with $q = 0.543$ ($\mu = 0.327$). The numbers of steps in the cascade n are, from top to bottom: 20.0, 18.2, 14.6, 11.4, 9.30, 7.90, 6.80, 5.60, 4.70 and 4.00.

of [21], whereas, the values of ℓ_c^{L}/η are different from the one obtained in [21] where Gotoh *et al* obtained the relation $\ell_c^{\mathrm{L}}/\eta \approx 2\ell_c^{\mathrm{T}}/\eta$ within their analyses. In spite of the difference in values of ℓ_c^{L}/η, the inequality $\ell_c^{\mathrm{T}}/\eta < \ell_c^{\mathrm{L}}/\eta$ may be attributed as a manifestation of the incompressible nature of the fluid under consideration, as interpreted in [21]. Notice that this kind of crossover was not observed in the analysis [19] of the experiment conducted by Lewis and Swinney [22] for turbulent Couette–Taylor flow.

The equations (13) and (15) have slopes close to -1, with respect to $\log_2 r/\eta$, which is consistent with (1) for $\delta = 2$. Therefore, we conclude that the *upper* scaling region $\ell_c^{\mathrm{L}}/\eta \leqslant r/\eta \lesssim 1220$ ($\ell_c^{\mathrm{T}}/\eta \leqslant r/\eta \lesssim 1220$) found within the present systematic analysis can be interpreted as the inertial range for longitudinal (transverse) fluctuations. Note that this inertial range for longitudinal fluctuations is wider than the longitudinal scaling range $2\lambda/\eta \lesssim r/\eta \lesssim 220.7$ observed in [21] at $R_\lambda = 381$. The latter estimate was obtained by looking for the flat region of the observed ζ_m^{L} with respect to the separation r/η. The existence of the crossover length was observed first by Gotoh *et al* [21] in the analyses of their DNS data as the low end of the scaling range for the longitudinal or the transverse fluctuations, and was attributed to the existence of structure of this order in the turbulence, e.g., a shear layer or a vortex tube.

It is remarkable that in figure 5 we can see, within the present self-consistent analysis, the existence of another new scaling region, the *lower* scaling region, i.e., $2 \lesssim r/\eta \leqslant \ell_c^{\mathrm{L}}/\eta$ ($2 \lesssim r/\eta \leqslant \ell_c^{\mathrm{T}}/\eta$) for longitudinal (transverse) velocity fluctuations. It may be of interest that the slopes in (14) and (16) can be -1 with respect to $\log_\delta r/\eta$, with $\delta = \delta^{\mathrm{L}} = 1.33$ for

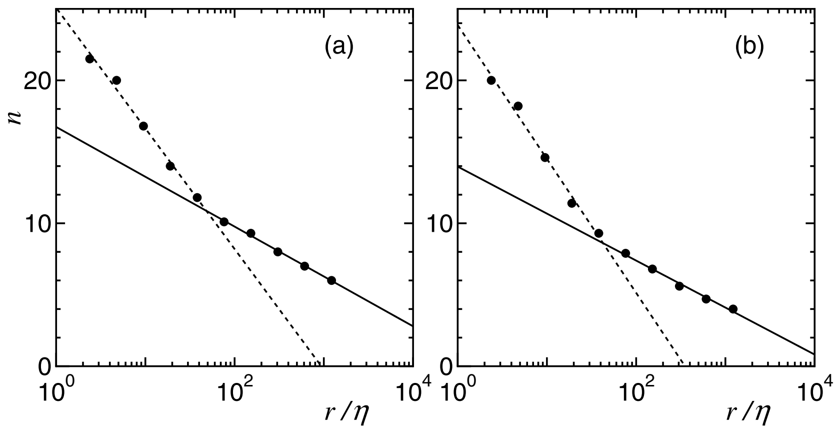

Figure 5. The dependence of n on r/η is shown by closed circles. (a) Those extracted from figure 3 for *longitudinal* fluctuations. The crossover occurs at $\ell_c^{\mathrm{L}}/\eta = 48.26$. (b) Those from figure 4 for *transverse* fluctuations. The crossover occurs at $\ell_c^{\mathrm{T}}/\eta = 42.57$.

longitudinal fluctuations and with $\delta = \delta^{\mathrm{T}} = 1.28$ for transverse ones. Therefore, in this *lower* scaling region, the interpretation is that eddies break up, effectively, into δ^{L} (δ^{T}) pieces at each step of the energy cascade for longitudinal (transverse) fluctuations.

Assuming that (1) with $\delta = 2$ is applicable in the 'inertial range' $\ell_c^{\mathrm{T}}/\eta, \ell_c^{\mathrm{L}}/\eta \leqslant r/\eta \lesssim \ell_0/\eta$, we can extract from (13) and (15) the values of the Reynolds number $Re^{\mathrm{L}} = 5.236 \times 10^6$, for longitudinal eddies, and $Re^{\mathrm{T}} = 7.113 \times 10^5$, for transverse eddies. If we adopt the relation $R_\lambda^2 = A^{\mathrm{L}}\, Re^{\mathrm{L}} = A^{\mathrm{T}}\, Re^{\mathrm{T}}$ [24], we have $A^{\mathrm{L}} = 0.027\,72$ and $A^{\mathrm{T}} = 0.2041$.

In this paper, we showed how precisely the formulae, derived using our theory [15–20], can explain the data observed in the DNS [21]. We found, explicitly, that there exist two scaling regions, i.e., the *upper* scaling region with larger separations which may correspond to the scaling range observed by Gotoh *et al*, and the *lower* scaling region with smaller separations which is a new scaling region extracted for the first time by the present systematic analysis. These scaling regions are divided by a definite length approximately of the order of the Taylor microscale λ, which may correspond to the crossover length introduced by Gotoh *et al* [21] as the length at which a scaling behaviour of the structure functions ceases due to the effect of kinematic viscosity. We also saw that the inertial range (the *upper* scaling region) for longitudinal fluctuations is wider than the scaling range extracted by Gotoh *et al* [21]. The discovery of the *lower* scaling region by the present analysis may tell us that the distribution of singularities of the velocity gradient in physical space is robust enough to produce a scaling behaviour even in the dissipation range. In other words, the system of turbulence may have an intrinsic scaling property, operative over a wider region than the inertial range, which can be attributed as arising from the multifractal distribution of the singularities in the velocity gradient. A study of the PDFs for velocity derivatives using the present analysis is an attractive open problem and is now in progress. One attractive future problem is that of obtaining a dynamical approach based on aspects of the present ensemble theoretical approach that can be used to interpret—in particular—the *lower* scaling region. The findings will be reported elsewhere.

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